

# RELATIVISTIC HYDRODYNAMICS - CAUSALITY AND STABILITY

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**ABSTRACT.** Causality and stability in relativistic dissipative hydrodynamics are important conceptual issues. We argue that causality is not restricted to hyperbolic set of differential equations. E.g. heat conduction equation can be causal considering the physical validity of the theory. Furthermore we propose a new concept of relativistic internal energy that clearly separates the dissipative and non-dissipative effects. We prove that with this choice we remove all known instabilities of the linear response approximation of viscous and heat conducting relativistic fluids. In this paper the Eckart choice of the velocity field is applied.

## 1. INTRODUCTION

The theories of dissipative fluids are different in the relativistic and nonrelativistic spacetime. The simplest first order, parabolic theory of nonrelativistic fluids, the system of Navier-Stokes and Fourier equations, is tested and justified by countless applications of the everyday engineering practice. The second order generalization of the theory introduces the fluxes of the extensives as independent variables. In this way the validity and applicability of the hydrodynamic and heat conduction equations is extended providing a hyperbolic system [1, 2]. Although the well known special relativistic generalization of the first order Navier-Stokes-Fourier system is straightforward [3], it has some unacceptable features. It is acausal and unstable. Therefore it is generally accepted that only the extended, second order theories are viable. Nowadays, heavy ion collision experiments give a unique opportunity to check the different suggestions and the interest in dissipative relativistic theories is renewed [4, 5], [6, 7, 8, 9, 10, 11, 12].

The first order theories are based on the local equilibrium hypothesis, where the independent variables are the same as in equilibrium, but in second (and higher) order theories the fluxes of the local equilibrium theory appear as independent variables. An other usual property is that in case second order relativistic theories the entropy vector is quadratic in the fluxes, containing terms like  $q_\alpha q^\alpha u^\mu$ ,  $\Pi^{\mu\alpha} q_\alpha$  etc., characterizing the deviation from the local equilibrium. As one can introduce general dynamic variables beyond the fluxes (see e.g. [13]), the above classification is not general.

Differential equations of first order theories are parabolic, therefore they are considered generally as acausal. The differential equations of the second order theories, that are constructed according to the Second Law, are mostly hyperbolic, therefore they are considered generally as causal. However, more careful considerations show, that the relation between parabolicity and causality is not so straightforward and

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requires some attention both from a mathematical and from a physical point of view.

The homogeneous equilibrium in first order theories is generally considered as unstable. The homogeneous equilibrium of second order theories is generally considered as stable. The stability considerations are referring to the linear stability calculations of Hiscock and Lindblom [14, 15].

However, like these second order theories are extending the validity of the first order ones, their physical content is included in the corresponding more general second order theory. The more involved second order theories do not cure necessarily the instabilities of the first order theories. It is shown by Geroch and Lindblom that physical fluid states in these theories relax to the solutions of the underlying first order theory [16, 17].

The missing of a simple and stable relativistic generalization of the Navier-Stokes-Fourier theory resulted in several attempts to improve the properties of first order theories. García-Colín and Sandoval-Villalazo suggested a separation of an internal energy balance from the balance of the energy-momentum, similarly to nonrelativistic theories [18]. However, with an additional independent energy balance the energy-momentum tensor would not embrace the whole energy content of the matter [19]. Other authors suggest a suitable definition of the four-velocity field [20]. None of the previous suggestions investigate the stability of the corresponding equations.

In the following section we argue that the speed of the propagation of signals can be finite in parabolic theories, too, if their physical validity is considered. Therefore first order relativistic hydrodynamic theories cannot be excluded by referring to causality. Moreover, in the light of the above mentioned observation of Geroch and Lindblom it is even more important to find a viable relativistic generalization of the Navier-Stokes-Fourier equations. A necessary condition for a causality in a weaker sense [21] is the stability of the homogeneous solutions of the corresponding differential equations. Based on this observation in the followings we outline a new approach to relativistic fluids. We suggest a separation of the dissipative and non-dissipative parts of the energy momentum distinguishing between the total energy density and the internal energy density of the matter. The later is the absolute value of the projected energy flux four-vector, this way it incorporates the momentum density as well. Since in the corresponding thermodynamic frame the entropy density depends also on the energy flux besides the energy density, but does not on the pressure, our suggestion can be classified between the first order, local equilibrium one, and of the extended, second order theories. In the final section we demonstrate the linear stability of the homogeneous equilibrium of viscous, heat conducting fluids with the Eckart choice of the velocity field.

## 2. REMARKS ON CAUSALITY

The common argument against the use of parabolic differential equations in physics is that some of their typical solutions show signal propagation with infinite speed. More sophisticated arguments require a well posedness of the related mathematical problems that can be guaranteed by hyperbolicity. The characteristic surface of the simplest relativistic heat conduction equation with constant coefficient is a spacelike hypersurface according to a comoving observer. Moreover, the characteristic surfaces are invariant to the transformation of the equation (in

particular to a Lorentz boost of the reference frame): The parabolicity or the hyperbolicity of the equation does not change by changing the observer. Therefore, as it was argued by Kostädt and Liu [22], by simple mathematics initial value problems of parabolic differential equations can be well posed, provided that initial data are given on the characteristic surface of the equations. From a physical point of view this is a natural requirement.

On the other hand, the characteristic hypersurfaces are those that determine the speed of propagation of simple solutions (the domain of influence) of a hyperbolic differential equation, too. Therefore, speed of the signal propagation for a hyperbolic differential equation is in general not infinite, but can be higher than the speed of light. The actual speed depends on the parameter values in the equation. This statement appears as a trivial fact in case of wave propagation equation. In one space dimension, considering a comoving observer, with respect to a constant velocity field  $u_\alpha$  we get the following form

$$(1) \quad \partial_{tt}\theta - c_w^2 \partial_{xx}\theta = 0.$$

Here  $\theta$  is the corresponding scalar physical quantity,  $c_w$  is the wave propagation speed. The solution of the characteristic differential equation of (1) gives the equation  $\theta(x, t) = x \pm c_w t = \text{const.}$  for the two characteristic lines of the equation. Applying a Lorentz transformation  $\tilde{x} = \gamma(x - vt)$  and  $\tilde{t} = \gamma(t - vx/c^2)$  we can get the transformed form of the characteristic lines as:

$$\theta(\tilde{x}, \tilde{t}) = (1 \pm \frac{c_w v}{c^2})x + (v \pm c_w)t.$$

Therefore the transformed characteristic speed is

$$\tilde{c}_w = \frac{v \pm c_w}{1 \pm \frac{c_w v}{c^2}}.$$

We can get the same result with the Lorentz transformation of the equation, too. The above expression shows that the propagation speed of waves can be faster than the speed of the light only if  $c_w > c$ , as we have expected.

In case of a set of nonlinear differential equations the calculation of the characteristic wave speeds can be more involved and even the proof of the hyperbolicity of the corresponding set of equations is not trivial. In general the value of the speed will depend on parameters in the set of equations and the relativistic, covariant form combined with hyperbolicity do not warrant a propagation speed smaller than the speed of light.

On the other hand the theoretically infinite speeds in parabolic equations are usually not observable, because their effect is out of the physical validity range of the theory. The atomistic structure of the matter restricts the validity of continuum descriptions as it was pointed out by Weymann [23, 22]. With the help of the mean free path and the collision time one can give simple estimates on the propagation speed of measurable signals.

We demonstrate this property on the example of the Fourier heat conduction equation

$$(2) \quad \partial_t \theta - \lambda \partial_{xx} \theta = 0.$$

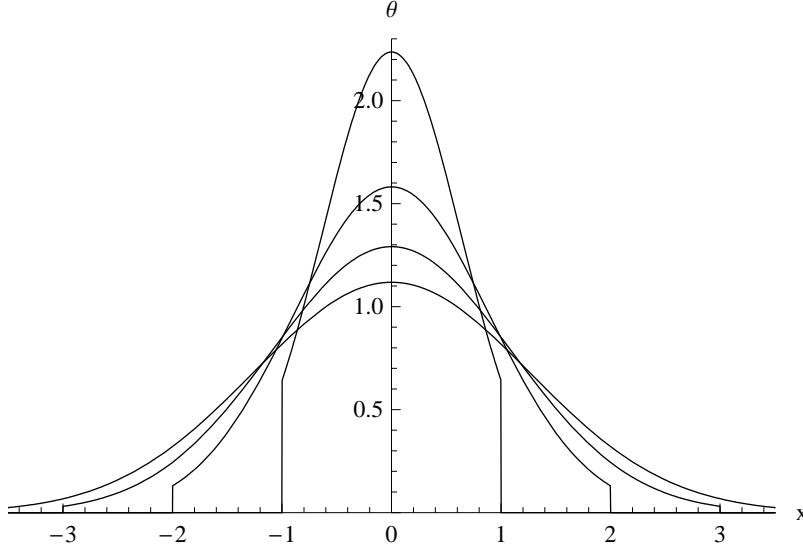


FIGURE 1. Mean free path limited signal propagation according to the Fourier heat conduction equation.  $\xi = .2$ ,  $\lambda = 1$  and  $t = 0.2, 0.4, 0.6, 0.8$

The hydrodynamic range of validity requires that  $\theta$  must not vary too rapidly over a mean free path  $\xi$

$$(3) \quad \left| \frac{1}{\theta} \frac{\partial \theta}{\partial x} \right| \ll \frac{1}{\xi}.$$

Assuming a sharp initial condition the solution of the heat conduction equation can be written as

$$\theta(x, t) = \frac{A}{\sqrt{2\pi t}} e^{-\frac{x^2}{4\lambda t}}$$

This is a typical acausal solution of the Fourier equation. However, substituting the above solution into the condition (3) we get a limit of the propagation speed of the continuum signals as

$$(4) \quad \frac{x}{t} \ll v_{lim} \propto \frac{\lambda}{\xi}.$$

Therefore, instead of the infinite tail of the solution in the reality we have an extending range, cf. (1).

In case of heat conduction in water at room temperature we can easily give an estimate as  $v_{max} \sim \frac{\Lambda}{c_v \rho \xi} \simeq 14m/s$ , where  $\Lambda$  is the Fourier heat conduction coefficient,  $\rho$  is the density and  $c_v$  is the specific heat of water. As heat conduction is disputed phenomena in quark-gluon plasma we cannot give a reliable estimation here.

Independently of the previous estimation Fichera suggested that the speed of the signal propagation is restricted by observability of the given physical quantity. Observability can be related to the sensitivity of the measurement but also to fluctuations and the particular structure of the matter. That can give an other bound to the speed of the signal propagation [24, 25, 21]. In our case we may

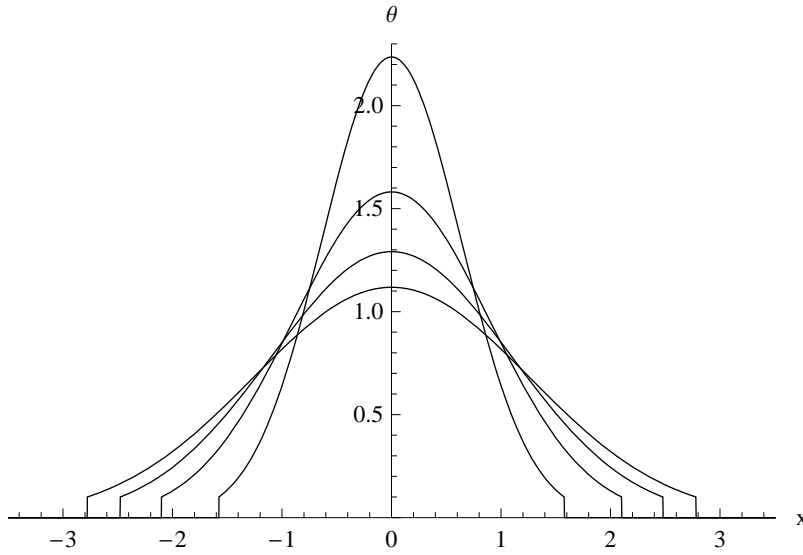


FIGURE 2. Signal propagation according to the observability threshold of the Fourier heat conduction equation.  $\theta_{max} = 0.1$ ,  $\lambda = 1$  and  $t = 0.2, 0.4, 0.6, 0.8$

assume that we cannot observe  $|\theta|$  below a given value  $|\theta| < \theta_{max}$ . Then the propagation speed becomes finite, nevertheless it is not constant as one can inspect in figure (2).

Summarizing our arguments have seen that hyperbolicity of the equations can lead to well posed problems and gives finite propagation speed but does not warrant that the propagation speed is less than the speed of light. On the other hand one can formulate well posed Cauchy problems related to parabolic equations, too. Moreover, the physical validity of a continuum theory can warrant slow propagation speeds in several different ways. Therefore, we may conclude that parabolic and mixed systems of continuum differential equations (as Fourier heat conduction or any first order continuum hydrodynamics) could be useful models in relativistic theories. Those theories cannot be excluded by causality arguments.

All the previous estimates are connected to some definite properties of the solutions of the simple heat conduction equation. If the exponential damping of the solutions cannot be guaranteed, then causality issues can become important. Therefore the stability of the homogeneous equilibrium is not only an evident physical requirement but also a necessary condition for the causality of any first order dissipative relativistic hydrodynamics.

### 3. BALANCES OF PARTICLE NUMBER, ENERGY AND MOMENTUM OF RELATIVISTIC FLUIDS

For the metric (Lorentz form) we use the  $g^{\mu\nu} = \text{diag}(-1, 1, 1, 1)$  convention and we use a unit speed of light  $c = 1$ , therefore for a four-velocity  $u^\alpha$  we have  $u_\alpha u^\alpha = -1$ .  $\Delta^\alpha_\beta = g^\alpha_\beta + u^\alpha u_\beta$  denotes the  $u$ -orthogonal projection. This metric convention will be convenient in the stability calculations.

In the following we fix the velocity field to the particle number flow, according to Eckart. Therefore the particle number flow is timelike by definition and can be expressed by the local rest frame quantities as

$$(5) \quad N^\alpha = nu^\alpha.$$

Here  $n = -u_\alpha N^\alpha$  is the *particle density* in comoving frame.

The particle number conservation is described by

$$(6) \quad \partial_\alpha N^\alpha = \dot{n} + n\partial_\alpha u^\alpha = 0,$$

where  $\dot{n} = \frac{dn}{d\tau} = u^\alpha \partial_\alpha n$  denotes the derivative of  $n$  with respect to the proper time  $\tau$ .

The energy-momentum density tensor is given with the help of the rest-frame quantities as

$$(7) \quad T^{\alpha\beta} = eu^\alpha u^\beta + u^\alpha q^\beta + u^\beta q^\alpha + P^{\alpha\beta},$$

where  $e = u_\alpha u_\beta T^{\alpha\beta}$  is the *density of the energy*,  $q^\beta = -u_\alpha \Delta^\beta_\gamma T^{\alpha\gamma}$  is the *energy flux* or *heat flux*,  $q^\alpha = -u_\beta \Delta^\alpha_\gamma T^{\gamma\beta}$  is the *momentum density* and  $P^{\alpha\beta} = \Delta^\alpha_\gamma \Delta^\beta_\mu T^{\gamma\mu}$  is the *pressure (stress) tensor*. The momentum density, the energy flux and the pressure are spacelike in the comoving frame, therefore  $u_\alpha q^\alpha = 0$  and  $u_\alpha P^{\alpha\beta} = u_\beta P^{\alpha\beta} = 0^\beta$ . Let us emphasize that the (7) form of the energy-momentum tensor is completely general, it is just expressed by the local rest frame quantities. The energy-momentum tensor is symmetric, because we assume that the internal spin of the material is zero. In this case the heat flux and the momentum density are equal. However, the difference in their physical meaning is a key element of our train of thoughts. Heat is related to dissipation of energy but momentum density is not, therefore this difference should appear in the corresponding thermodynamic framework.

Now the conservation of energy-momentum  $\partial_\beta T^{\alpha\beta} = 0$  is expanded to

$$(8) \quad \partial_\beta T^{\alpha\beta} = \dot{e}u^\alpha + eu^\alpha \partial_\beta u^\beta + e\dot{u}^\alpha + u^\alpha \partial_\beta q^\beta + q^\beta \partial_\beta u^\alpha + \dot{q}^\alpha + q^\alpha \partial_\beta u^\beta + \partial_\beta P^{\alpha\beta}.$$

Its timelike part in the local rest frame gives the balance of the energy  $e$

$$(9) \quad -u_\alpha \partial_\beta T^{\alpha\beta} = \dot{e} + e\partial_\alpha u^\alpha + \partial_\alpha q^\alpha + q^\alpha \dot{u}_\alpha + P^{\alpha\beta} \partial_\beta u_\alpha = 0.$$

The spacelike part in the local rest frame describes the balance of the momentum

$$(10) \quad \Delta^\alpha_\gamma \partial_\beta T^{\gamma\beta} = e\dot{u}^\alpha + q^\alpha \partial_\beta u^\beta + q^\beta \partial_\beta u^\alpha + \Delta^\alpha_\gamma \dot{q}^\gamma + \Delta^\alpha_\gamma \partial_\beta P^{\beta\gamma} = 0.$$

#### 4. THERMODYNAMICS

The entropy density and flux can also be combined into a four-vector, using local rest frame quantities:

$$(11) \quad S^\alpha = su^\alpha + J^\alpha,$$

where  $s = -u_\alpha S^\alpha$  is the *entropy density* and  $J^\alpha = S^\alpha - u^\alpha s = \Delta^\alpha_\beta S^\beta$  is the *entropy flux*. The entropy flux is  $u$ -spacelike, therefore  $u_\alpha J^\alpha = 0$ . Now the Second Law of thermodynamics is translated to the following inequality

$$(12) \quad \partial_\alpha S^\alpha = \dot{s} + s\partial_\alpha u^\alpha + \partial_\alpha J^\alpha \geq 0$$

Relativistic thermodynamic theories assume that the entropy is a function of the local rest frame quantities, because the thermodynamic relations reflect general properties of local material interactions. The most important assumption is that the entropy is a function of the local rest frame energy density, the time-timelike

component of the energy momentum tensor according to the velocity field of the material [3, 26]. Definitely the thermodynamics cannot be related to an external observer, therefore the dependence on the relative kinetic energy is excluded. This interpretation of  $e$  in (7) is supported by the form of the energy balance (9), where the last term is analogous to the corresponding internal energy source (dissipated power) of the nonrelativistic theories.

In nonrelativistic fluids the internal energy is the difference of the conserved total energy and the kinetic energy of the material. However, also in nonrelativistic theories the constitutive relations must be objective in the sense that they cannot depend on an external observer, the thermodynamic framework should produce frame independent material equations. (This apparent contradiction of classical physics is eliminated by different sophisticated methods and lead to such important concepts as the configurational forces or/and virtual power [27, 28, 29]). However, without distinguishing the energy related to the flow of the material from the total energy one mixes the dissipative and nondissipative effects. The wrong separation leads to generic instabilities of the corresponding theory.

Our candidate of the relativistic internal energy is related to the energy vector defined by  $E^\alpha = -u_\beta T^{\alpha\beta} = eu^\alpha + q^\alpha$ . The energy vector embraces both the total rest frame energy density and the rest frame momentum. Therefore its absolute value  $\|E\| = \sqrt{-E_\alpha E^\alpha} = \sqrt{|e^2 - q_\alpha q^\alpha|}$  seems to be a reasonable choice of the scalar internal energy<sup>1</sup>. Its series expansion, when the energy density is larger than the momentum density, shows strong analogies to the corresponding nonrelativistic definition

$$\|E\| = \sqrt{|e^2 - q_\alpha q^\alpha|} \approx e - \frac{\mathbf{q}^2}{2e} + \dots$$

Thermodynamic calculations based on the Liu procedure support this assumption [30]. Let us emphasize, that our candidate of internal energy is not related to any external reference frame, only to the velocity field of the material. In a Landau-Lifschitz frame the energy vector is timelike.

Assuming that the entropy density is the function of the internal energy and the particle number density  $s(e, q^\alpha, n) = \hat{s}(\sqrt{|e^2 - q_\alpha q^\alpha|}, n)$  leads to a modified form of the thermodynamic Gibbs relation and the potential relation for the densities as follows

$$(13) \quad de - \frac{q^\alpha}{e} dq_\alpha = T ds + \mu dn, \quad \text{and} \quad e - \frac{q^2}{e} = Ts - p + \mu n.$$

Here  $T$  is the temperature,  $p$  is the pressure and  $\mu$  is the chemical potential. Equivalently the Gibbs relation gives the derivatives of the entropy density as follows

$$\left. \frac{\partial s}{\partial e} \right|_{(q^\alpha, n)} = \frac{1}{T}, \quad \left. \frac{\partial s}{\partial q^\alpha} \right|_{(e, n)} = -\frac{q_\alpha}{eT}, \quad \left. \frac{\partial s}{\partial n} \right|_{(q^\alpha, e)} = -\frac{\mu}{T}.$$

For the entropy flux we assume the classical form

$$(14) \quad J^\alpha = \frac{q^\alpha}{T}.$$

Now we substitute the energy balance (9) and the particle number balance (6) into the entropy balance (12) and we arrive at the following entropy production

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<sup>1</sup>Note that  $q^\alpha$  is spacelike, therefore the quantity under the sign is non-negative.

formula:

$$\begin{aligned}
\partial_\alpha S^\alpha &= \dot{s}(e, q^\alpha, n) + s\partial_\alpha u^\alpha + \partial_\alpha J^\alpha \\
&= \frac{\partial s}{\partial e}\dot{e} + \frac{\partial s}{\partial q^\alpha}\dot{q}^\alpha + \frac{\partial s}{\partial n}\dot{n} + s\partial_\alpha u^\alpha + \partial_\alpha \frac{q^\alpha}{T} \\
&= -\frac{1}{T}(e\partial_\alpha u^\alpha + \partial_\alpha q^\alpha + q^\alpha \dot{u}_\alpha + P^{\alpha\beta}\partial_\beta u_\alpha) - \frac{q^\alpha}{Te}\dot{q}_\alpha + s\partial_\alpha u^\alpha \\
&\quad + \frac{\mu}{T}n\partial_\alpha u^\alpha + q^\alpha\partial_\alpha \frac{1}{T} + \frac{1}{T}\partial_\alpha q^\alpha \\
(15) \quad &= -\frac{1}{T}(P^{\alpha\beta} - (-e + sT + \mu n)\Delta^{\alpha\beta})\partial_\alpha u_\beta + q^\alpha\left(\partial_\alpha \frac{1}{T} - \frac{\dot{u}_\alpha}{T} - \frac{\dot{q}_\alpha}{eT}\right) \geq 0.
\end{aligned}$$

According to this quadratic expression and the potential relation in (13) the *viscous pressure* is given by

$$\Pi^{\alpha\beta} = P^{\alpha\beta} - \left(p - \frac{q^2}{e}\right)\Delta^{\alpha\beta}$$

We may introduce the conventional decomposition of the pressure

$$\Pi^{\alpha\beta} = (p + \Pi)\Delta^{\alpha\beta} + \langle \Pi^{\alpha\beta} \rangle.$$

Here  $\Pi = \frac{1}{3}\Pi^\alpha_\alpha - p = \frac{1}{3}P^\alpha_\alpha - \frac{q^2}{e}$  and  $\langle \Pi^{\alpha\beta} \rangle = \Delta^\alpha_\mu \Delta^\beta_\nu (\frac{1}{2}(\Pi^{\mu\nu} + \Pi^{\nu\mu}) - \frac{1}{3}\Delta^{\mu\nu}\Delta_{\gamma\delta}\Pi^{\gamma\delta})$  is symmetric and traceless.

Therefore the (15) entropy production can be written as

$$(16) \quad -\frac{1}{T}\langle \Pi^{\alpha\beta} \rangle \partial_\alpha u_\beta - \frac{1}{T}\left(\Pi - \frac{q^2}{e}\right)\partial_\alpha u^\alpha + q^\alpha\left(\partial_\alpha \frac{1}{T} - \frac{\dot{u}_\alpha}{T} - \frac{\dot{q}_\alpha}{eT}\right) > 0$$

In isotropic continua the above entropy production results in the following constitutive functions assuming a linear relationship between thermodynamic fluxes and forces

$$(17) \quad q^\alpha = -\tilde{\lambda}\frac{1}{T^2}\Delta^{\alpha\gamma}\left(\partial_\gamma T + T\dot{u}_\gamma + \frac{\dot{q}_\gamma}{e}\right),$$

$$(18) \quad \langle \Pi^{\alpha\beta} \rangle = -2\eta\langle \partial^\alpha u^\beta \rangle,$$

$$(19) \quad \Pi - \frac{q^2}{e} = -\eta_v\partial_\alpha u^\alpha.$$

Let us recognize the additional term to the bulk viscous pressure. For the whole viscous stress we get

$$(20) \quad \Pi^{\alpha\beta} = -2\eta\langle \partial^\alpha u^\beta \rangle - \eta_v\Delta^{\alpha\beta}\partial_\gamma u^\gamma.$$

(17) and (20) are the relativistic generalizations of the Fourier law of heat conduction and the Newtonian viscous pressure function. The shear and bulk viscosity coefficients,  $\eta$  and  $\eta_v$  and the heat conduction coefficient  $\lambda = \tilde{\lambda}T^{-2}$  are non negative, according to the inequality of the entropy production (17). We may introduce a relaxation time  $\tau = \lambda/e$  in (17), as usual in second order theories.

The equations (6) and (9) are the evolution equations of a relativistic heat conducting ideal fluid, together with the constitutive function (20) and the relaxation type equation (17). As special cases we can get the relativistic Navier-Stokes equation substituting (20) into (10) and assuming  $q^\alpha = 0$ , or the equations of relativistic heat conduction solving together (17) and (9) assuming that  $\Pi^{\alpha\beta} = 0$  and  $u^\alpha = \text{const.}$



## 5. LINEAR STABILITY

In this section we investigate the linear stability of the homogeneous equilibrium of the equations (6), (9) and (10) together with the constitutive relations (20) and (17). Similar calculations are given by Hiscock and Lindblom both for Eckart fluids [14] and Israel-Stewart fluids [15].

**5.1. Equilibrium.** The equilibrium of the above set of equations is defined by vanishing proper time derivatives and by zero entropy production with vanishing thermodynamic fluxes

$$(21) \quad \Pi^{\alpha\beta} = 0 \quad \text{and} \quad q^a = 0.$$

Therefore according to the balances and the constitutive functions the equilibrium of the fluid is determined by

$$(22) \quad n = \text{const.} \quad e = \text{const.} \quad \Rightarrow \quad T = \text{const.}, \quad \mu = \text{const.}, \quad p = \text{const.},$$

$$(23) \quad \partial_\alpha u_\alpha = 0, \quad \partial_\alpha u_\beta + \partial_\beta u_\alpha = 0.$$

In addition to the above conditions we require a homogeneous equilibrium velocity field

$$(24) \quad u_\alpha = \text{const.}$$

**5.2. Linearization.** We denote the equilibrium fields by zero lower index and the perturbed fields by  $\delta$  as  $Q = Q_0 + \delta Q$ . Here  $Q$  stands for  $n$ ,  $e$ ,  $u^\alpha$ ,  $q^\alpha$ , and  $\Pi^{\alpha\beta}$ . The linearized equations (6), (9), (10), (17), (20) around the equilibrium given by (21)-(22)-(24) become

$$(25) \quad 0 = \dot{\delta n} + n \partial_\alpha \delta u^\alpha,$$

$$(26) \quad 0 = \dot{\delta e} + (e + p) \partial_\alpha \delta u^\alpha + \partial_\alpha \delta q^\alpha,$$

$$(27) \quad 0 = (e + p) \delta \dot{u}^\alpha + \Delta^{\alpha\beta} \partial_\beta \delta p + \delta \dot{q}^\alpha + \Delta_\gamma^\alpha \partial_\beta \delta \Pi^{\gamma\beta},$$

$$(28) \quad 0 = \delta q^\alpha + \lambda \Delta^{\alpha\gamma} \left( \partial_\gamma \delta T + T \delta \dot{u}_\gamma + \frac{T}{e} \delta \dot{q}_\gamma \right),$$

$$(29) \quad 0 = \delta \Pi^{\alpha\beta} + \tilde{\eta}_v \partial_\gamma \delta u^\gamma \Delta^{\alpha\beta} + \eta \Delta^{\alpha\gamma} \Delta^{\beta\mu} (\partial_\gamma \delta u_\mu + \partial_\mu \delta u_\gamma).$$

Here  $\tilde{\eta}_v = (\eta_v - \frac{2}{3}\eta)$ . The perturbation variables satisfy the following properties inherited from the linearization of the original ones

$$0 = u^\alpha \delta q_\alpha = u^\alpha \delta u_\alpha = u^\alpha \delta \Pi_{\alpha\beta} = \delta \Pi_{\alpha\beta} - \delta \Pi_{\beta\alpha}$$

In order to identify possible instabilities we select out exponential plane-wave solutions of the perturbation equations:  $\delta Q = Q_0 e^{\Gamma t + i k x}$ , where  $Q_0$  is constant and  $t$  and  $x$  are two orthogonal coordinates in Minkowski spacetime. As our equilibrium background state is a fluid at rest we put  $u^\alpha \partial_\alpha = \partial_t$ .

With these assumptions the set of perturbation equations follow as

$$\begin{aligned}
0 &= \Gamma \delta n + ikn \delta u^x, \\
0 &= \Gamma \delta e + (e+p)ik \delta u^x + ik \delta q^x, \\
0 &= \Gamma(e+p) \delta u^x + ik(\partial_e p \delta e + \partial_n p \delta n) + \Gamma \delta q^x + ik \delta \Pi^{xx}, \\
0 &= \Gamma(e+p) \delta u^y + \Gamma \delta q^y + ik \delta \Pi^{xy}, \\
0 &= \Gamma(e+p) \delta u^z + \Gamma \delta q^z + ik \delta \Pi^{xz}, \\
0 &= \delta q^x + ik\lambda(\partial_e T \delta e + \partial_n T \delta n) + \lambda T \Gamma \delta u^x + \lambda \frac{T}{e} \Gamma \delta q^x, \\
0 &= \delta q^y + \lambda T \Gamma \delta u^y + \lambda \frac{T}{e} \Gamma \delta q^y, \\
0 &= \delta q^z + \lambda T \Gamma \delta u^z + \lambda \frac{T}{e} \Gamma \delta q^z, \\
0 &= \delta \Pi^{xx} + ik \tilde{\eta} \delta u^x, \\
0 &= \delta \Pi^{xy} + ik \eta \delta u^y, \\
0 &= \delta \Pi^{xz} + ik \eta \delta u^z, \\
0 &= \delta \Pi^{yy} + ik \tilde{\eta}_v \delta u^y, \\
0 &= \delta \Pi^{zz} + ik \tilde{\eta}_v \delta u^z, \\
(30) \quad 0 &= \delta \Pi^{zy}.
\end{aligned}$$

Here we have introduced a shortened notation for  $\tilde{\eta} = \eta_v + \frac{4}{3}\eta$ . We can put the equations above into the following matrix form

$$(31) \quad M^A_B \delta Q^B = 0.$$

Here  $\delta Q^B$  represents the list of fields which describe the perturbation of the fluid:

$$\begin{aligned}
\delta Q = (\delta n, \delta e, \delta u^x, \delta q^x, \delta \Pi^{xx}, \delta u^y, \delta q^y, \delta \Pi^{xy}, \delta \Pi^{yy}, \\
\delta u^z, \delta q^z, \delta \Pi^{xz}, \delta \Pi^{zz}, \delta \Pi^{yz}).
\end{aligned}$$

Then the 14x14 matrix  $\mathbf{M}$  can be written in the block diagonal form

$$(32) \quad \mathbf{M} = \begin{pmatrix} \mathbf{N} & 0 & 0 & 0 \\ 0 & \mathbf{R} & 0 & 0 \\ 0 & 0 & \mathbf{R} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where the submatrices  $\mathbf{R}$  and  $\mathbf{N}$  are defined as follows

$$(33) \quad \mathbf{R} = \begin{pmatrix} (e+p)\Gamma & \Gamma & ik & 0 \\ \lambda \Gamma T & 1 + \lambda \Gamma \frac{T}{e} & 0 & 0 \\ ik\eta & 0 & 1 & 0 \\ ik\tilde{\eta}_v & 0 & 0 & 1 \end{pmatrix},$$

$$(34) \quad \mathbf{N} = \begin{pmatrix} \Gamma & 0 & ikn & 0 & 0 \\ 0 & \Gamma & ik(e+p) & ik & 0 \\ ik\partial_n p & ik\partial_e p & \Gamma(e+p) & \Gamma & ik \\ ik\lambda\partial_n T & ik\lambda\partial_e T & \lambda \Gamma T & 1 + \lambda \Gamma \frac{T}{e} & 0 \\ 0 & 0 & ik\tilde{\eta} & 0 & 1 \end{pmatrix}.$$

Exponentially growing plane-wave solutions of (31) emerge whenever  $\Gamma$  and  $k$  satisfy the dispersion relation

$$(35) \quad \det \mathbf{M} = (\det \mathbf{N})(\det \mathbf{R})^2 = 0$$

with a positive real  $\Gamma$ . The roots of this equation are the roots obtained by setting the determinants of either  $\mathbf{N}$  or  $\mathbf{R}$  to zero.

The determinant of  $\mathbf{R}$  gives the condition

$$\lambda T \frac{p}{e} \Gamma^2 + \left( e + p + k^2 \tilde{\eta} \lambda \frac{T}{e} \right) \Gamma + \tilde{\eta} k^2 = 0.$$

The real parts of the roots of this polynomial are negative because the coefficients of both the linear and the quadratic term are positive.

The determinant of  $\mathbf{N}$  gives the following dispersion relation

$$\begin{aligned} \lambda p \frac{T}{e} \Gamma^4 + \left( e + p + k^2 \tilde{\eta} \lambda \frac{T}{e} \right) \Gamma^3 + k^2 \left( \tilde{\eta} + \lambda \frac{T}{e} (n \partial_n p + p \partial_e p) - \lambda n \partial_n T \right) \Gamma^2 + \\ k^2 \left( (e + p) \partial_e p + n \partial_n p + k^2 \tilde{\eta} \lambda \partial_e T \right) \Gamma + k^4 \lambda n (\partial_e T \partial_n p - \partial_e p \partial_n T) = 0. \end{aligned}$$

According to the Routh-Hurwitz criteria [31], the real parts of the roots of a fourth order polynomial  $a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 = 0$  are negative whenever

$$\begin{aligned} a_0 &> 0, \\ a_1 &> 0, \\ a_1 a_2 - a_0 a_3 &> 0, \\ (36) \quad (a_1 a_2 - a_0 a_3) a_2 - a_4 a_1^2 &> 0. \end{aligned}$$

We can see, that the first two conditions of (36) are fulfilled according to the Second Law, the nonnegativity of the entropy production.

Let us recall the conditions of thermodynamic stability

$$(37) \quad \partial_e T > 0,$$

$$(38) \quad \partial_n \frac{\mu}{T} > 0,$$

$$(39) \quad \partial_e T \partial_n \frac{\mu}{T} - \partial_n T \partial_e \frac{\mu}{T} \geq 0,$$

and the following useful identities

$$T \partial_e p = (e + p) \partial_e T + n T^2 \partial_e \frac{\mu}{T},$$

$$T \partial_n p = (e + p) \partial_n T + n T^2 \partial_n \frac{\mu}{T}.$$

Now, the third condition can be written in a simplified form as

$$\begin{aligned} a_1 a_2 - a_0 a_3 &= k^2 \lambda (n T)^2 \partial_n \frac{\mu}{T} + k^4 \lambda \tilde{\eta}^2 \frac{T}{e} + k^2 \tilde{\eta} (e + p) + \\ (40) \quad k^4 \tilde{\eta} \lambda^2 \frac{T}{e^2} \left( \partial_e T p^2 + 2 n \partial_n T p + T^2 n^2 \partial_n \frac{\mu}{T} \right) &\geq 0. \end{aligned}$$

The first three terms in the expression are positive. In the parenthesis of the last term we can recognize a second order polynomial of  $p$ . The discriminant of that polynomial is negative

$$(41) \quad D_1 = (2 n \partial_n T)^2 - 4 \partial_e T n^2 T^2 \partial_n \frac{\mu}{T} = -4 n^2 \left( \partial_e T \partial_n \frac{\mu}{T} - \partial_n T \partial_e \frac{\mu}{T} \right) < 0,$$

because of the last condition of thermodynamic stability. Therefore the expression in the parenthesis is positive for all  $p$ .

Hence the fourth condition of (36) expands to the following form

$$\begin{aligned}
& (a_1 a_2 - a_0 a_3) a_2 - a_4 a_1^2 = \\
& k^4 (e + p) \frac{\eta}{T} \left( \partial_e T (e + p)^2 + 2n \partial_n T (e + p) + T^2 n^2 \partial_n \frac{\mu}{T} \right) + \\
& \lambda k^4 \frac{n^2}{T^2} \left( \partial_n T (e + p) + \partial_n \frac{\mu}{T} n T^2 \right)^2 + \\
& \lambda k^6 \frac{\eta^2}{e} \left( e(e + p) \partial_e T + \partial_e T (e + p)^2 + 2n \partial_n T (e + p) + T^2 n^2 \partial_n \frac{\mu}{T} \right) + \\
& \lambda^2 k^6 \frac{\eta}{e^2} (p(e + p) \partial_e T + n^2 T^2 \partial_n \frac{\mu}{T}) \left[ (p(e + p) \partial_e T + (e + 2p) \partial_n T)^2 + \right. \\
& \left. n^2 e(e + 2p) (\partial_n T)^2 + n^2 T^2 \partial_n \frac{\mu}{T} \left( 2p^2 \partial_e T + n^2 T^2 \partial_n \frac{\mu}{T} + 2n(e + 2p) \partial_n T \right) \right] \\
& \lambda^2 k^8 \eta^3 \frac{T}{e} \partial_e T + \lambda^3 k^8 \eta^2 \partial T e^2 (n \partial_n T + p \partial_e T)^2 > 0.
\end{aligned}$$

In the first and third term we recognize the same polynomial expression of  $(e + p)$  as in (40) for  $p$ . Therefore all terms are clearly positive, only the term in the rectangular parenthesis requires separate investigation. We may recognize that it is a second order polynomial of  $e$ , with the discriminant

$$D = -4n^2 (n \partial_n T + p \partial_e T)^2 \left( 2p^2 (\partial_e T \partial_n \frac{\mu}{T} T^2 - (\partial_n T)^2) - (p \partial_n T + n T^2 \partial_n \frac{\mu}{T})^2 \right) < 0.$$

The coefficient of the  $e^2$  term is  $\partial_e T p^2 + 2n \partial_n T p + T^2 n^2 \partial_n \frac{\mu}{T} > 0$  is a positive quantity according to (41). Therefore the term in the rectangular parenthesis is positive, too.

We conclude that the homogeneous equilibrium of the relativistic heat conducting viscous relativistic fluids is stable in contrast to the corresponding equations of an Eckart fluid. We did not need to exploit any special additional stability conditions beyond the well known thermodynamic inequalities and the stability conditions of fluids. This is in strong contrast to the Israel-Stewart theory, where one should assume additional conditions [15].

## 6. CONCLUSION

In this paper we addressed causality and stability, the two most important conceptual issues in relativistic hydrodynamics.

We have collected arguments that in dissipative first order relativistic fluids applied to heavy ion collisions acausality related problems may be beyond the range of validity of the theory. Therefore first order theories, parabolic and mixed parabolic hyperbolic equations can be useful physical models in relativistic hydrodynamics.

Moreover, with a proper distinction of the total and internal energies we suggested a simple modification of the Eckart theory and we proved that its homogeneous equilibrium is stable by linear perturbations in case of the Eckart choice of the velocity field. With the Landau-Lifshitz form of the velocity field our theory simplifies to the Eckart theory, therefore the corresponding homogeneous equilibrium is stable, too [22]. However, the Landau-Lifshitz convention has some other undesirable properties, that we intend to discuss in a consequent paper.

The suggested relativistic form of the internal energy depends on the momentum density, therefore the entropy function is the function of the momentum density as well. Moreover, in the first approximation we have a regular second order theory with only one additional quadratic term in the entropy four vector. However, there was no need to introduce additional parameters, the coefficient of the quadratic term, and therefore the relaxation time in the generalized Fourier equation, is fixed. It was very important, that the general entropy vector is not a simple quadratic function in the heat flux. We have got a correction to the viscous bulk pressure, too.

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